Summary of the Ph.D. Thesis

Globally Rigid Frameworks and Rigid Tensegrity Graphs in the Plane

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1 Introduction

The field of combinatorial rigidity is centered around geometric properties of straight line realizations of graphs in $\mathbb{R}^d$ that can be derived from the combinatorial properties of the graph when the placement of vertices is general enough.

One interesting question about realizations is the uniqueness of the distance of pairs of vertices given that the lengths of the edges remain the same. In other words, is there a realization with the same edge lengths but different distance of two designated vertices? The answer can be different for different realizations, but in certain cases the answer is always no when the edge lengths are generic enough (i.e. algebraically independent over $\mathbb{Q}$). The first part of the dissertation characterizes this case for two important graph families and formulates a conjecture for the general case. This field has important applications in sensor network localization [3, 5].

If the distance of each vertex pair is determined by the edge lengths, then the realization is said to be unique, or globally rigid. It is known [8] when a graph with generic edge lengths is globally rigid in the plane. However, the problem becomes NP-hard [13], even for the one dimensional case, if the realization can be arbitrary. Moreover, there is no 'simple' sufficient condition for the global rigidity of a non-generic realization. Therefore the problem of algorithmically constructing a realization that is globally rigid is non-trivial even when we know that such a realization exists. The second part of the dissertation describes an algorithm for the construction of globally rigid realizations in the plane.

Another geometric property is rigidity, which means that the realization can not be continuously deformed with keeping the edge lengths constant other than by congruences of the whole space. Rigidity in the plane is a well understood problem, and depends only on the graph structure if the realization is not 'degenerate'. The problem becomes more interesting, however, if we allow certain edges to become longer and other edges to become shorter during the deformation. Tensegrity graphs are edge-labeled graphs that encode these restrictions. The third part of the dissertation focuses on the existence of rigid realizations of such tensegrity graphs in the plane.

A common theme in the proofs and algorithms presented in this work is that they all use some constructive characterization result of certain graph families. These results state that each member of the graph family can be constructed from a small graph using certain simple graph operations and that all graphs constructed this way belong to the graph family. Therefore one of the key elements in the proofs is that these operations on graphs or realizations preserve some rigidity property.

With the exception of one graph extension result on global rigidity that is stated generally for $\mathbb{R}^d$, all of the results in this work are for the $d = 2$ case. We note that for $d = 1$, almost all rigidity problems are trivial or easy, while for $d \geq 3$ they seem to be hopelessly hard.

In the rest of the introduction we will give the precise definition of rigidity concepts that are needed to formulate the main results of the dissertation.

A d-dimensional *framework* is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to $\mathbb{R}^d$. This p is called a d-dimensional *realization* of G. An alternative way to look at p is as an n-tuple of d-dimensional vectors, or equivalently, an nd-dimensional vector, $p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^{nd}$, also called a d-dimensional *configuration*.

Two frameworks (G, p) and (G, q) are *equivalent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $uv \in E$. Frameworks (G, p) and (G, q) are *congruent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $u, v \in V$.

We say that a framework (G, p) is *rigid* if there exists an $\epsilon > 0$ such that if (G, q) is equivalent to (G, p) and $||p(v) - q(v)|| < \epsilon$ for all $v \in V$ then (G, q) is congruent to (G, p).

An *infinitesimal motion* of a framework (G, p) is an assignment of infinitesimal velocities to the vertices, $q : V \to \mathbb{R}^d$ satisfying

$$(p(u) - p(v))(q(u) - q(v)) = 0 \tag{1}$$

for all pairs u, v with $uv \in E$. If we think of infinitesimal motions as nd-dimensional vectors, then the set of infinitesimal motions of a framework (G, p) is a linear subspace of $\mathbb{R}^{nd}$, given by the $|E|$ linear equations of the form (1). The matrix of this system of linear equations is the *rigidity matrix* of (G, p) and it is denoted by $R(G, p)$. This is a matrix of size $|E| \times nd$, where, for each edge $uv \in E$, in the row corresponding to uv, the entries in the d columns corresponding to vertices u and v contain the d coordinates of $(p(u) - p(v))$ and $(p(v) - p(u))$, respectively, and the remaining entries are zeros. With this notation, a vector $q \in \mathbb{R}^{nd}$ is an infinitesimal motion of (G, p) if and only if $R(G, p)q = 0$, in other words, the space of infinitesimal motions of (G, p) is the kernel of $R(G, p)$.

Let S be a $d \times d$ antisymmetric matrix and $t \in \mathbb{R}^d$. A *trivial infinitesimal motion* of (G, p) has the form $q(v) = Sp(v) + t$, for all $v \in V$. It is easy to see that these are indeed infinitesimal motions. A framework (G, p) is said to be *infinitesimally flexible* if it has a non-trivial infinitesimal motion, otherwise it is called *infinitesimally rigid*.

It is known [4] that if a framework (G, p) is infinitesimally rigid, then it is rigid. The converse of this is not true, a framework can be rigid, but not infinitesimally rigid. However, if we exclude certain 'degenerate' configurations, rigidity and infinitesimal rigidity will become equivalent. A configuration $p \in \mathbb{R}^{nd}$ is a *regular point* of G if $\operatorname{rank} R(G,p) = \max\{\operatorname{rank} R(G,q) : q \in \mathbb{R}^{dn}\}$. A framework (G, p) is *regular* if p is a regular point of G. The set of regular points of G is an open dense subset of $\mathbb{R}^{nd}$ and if a regular framework (G, p) is infinitesimally rigid, then all other regular frameworks (G, q) will be infinitesimally rigid as well. Moreover, rigidity and infinitesimal rigidity is equivalent for regular frameworks.

We say that the graph G is *rigid in* $\mathbb{R}^d$, if every (or equivalently, if some) regular d-dimensional framework (G, p) is rigid (or equivalently, infinitesimally rigid).

Let (G, p) be a d-dimensional realization of a graph $G = (V, E)$. The rigidity matrix of (G, p) defines the *rigidity matroid* $\mathcal{R}_d(G, p)$ of (G, p) on the ground set E by linear independence of rows of the rigidity matrix. We say that a framework (G, p) is *strongly regular* if (H, p) is regular for all subgraphs H of G. Any two strongly regular frameworks (G, p) and (G, q) have the same rigidity matroid. We call this the *rigidity matroid* $\mathcal{R}_d(G) = (E, r)$ of the graph G. We denote the rank of $\mathcal{R}_d(G)$ by $r_d(G)$. We say that a graph $G = (V, E)$ is *M-independent in* $\mathbb{R}^d$ if E is independent in $\mathcal{R}_d(G)$. A graph $G = (V, E)$ is *minimally rigid in* $\mathbb{R}^d$ if G is rigid in $\mathbb{R}^d$, but $G - e$ is not rigid for all $e \in E$. A graph $G = (V, E)$ is *redundantly rigid in* $\mathbb{R}^d$ (a framework (G, p) is redundantly rigid) if $G - e$ is rigid in $\mathbb{R}^d$ (the framework $(G - e, p)$ is infinitesimally rigid) for all $e \in E$.

We say that a framework (G, p) is *globally rigid* if every framework (G, q) which is equivalent to (G, p) is congruent to (G, p). Unlike infinitesimal rigidity, which can be decided in polynomial time by checking the rank of the rigidity matrix, Saxe [13] has shown that it is NP-hard to decide if even a 1-dimensional framework is globally rigid. The problem becomes more tractable, however, if we assume that there are no algebraic dependencies between the coordinates of the points of the framework. A framework (G, p) (or a configuration $p \in \mathbb{R}^{nd}$) is said to be *generic* if the set containing the coordinates of all its points is algebraically independent over $\mathbb{Q}$.

A necessary condition for the global rigidity of a generic framework is from Hendrickson.

Theorem 1.1. *[7] Let (G, p) be a d-dimensional generic framework. If (G, p) is globally rigid then either G is a complete graph with at most $d+1$ vertices, or G is $(d+1)$-connected and (G, p) is redundantly rigid.*

It is an interesting question whether global rigidity is a generic property of a graph G in the sense that if a generic framework (G,p) is globally rigid, is it true that every other generic framework (G,q) is globally rigid as well? A positive answer to this question was given in [6], therefore we can say that a graph G is *globally rigid* in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is globally rigid.

2 Globally linked pairs of vertices

The results of this section are based on [9, 14]. A pair of vertices $\{u,v\}$ in a framework (G,p) is *globally linked*, if, in all equivalent frameworks (G,q), we have $||p(u)-p(v)|| = ||q(u)-q(v)||$. The pair $\{u,v\}$ is *globally linked* in G in $\mathbb{R}^d$ if it is globally linked in all d-dimensional generic frameworks (G,p). Thus G is globally rigid in $\mathbb{R}^d$ if and only if all pairs of vertices of G are globally linked in $\mathbb{R}^d$. Unlike global rigidity, however, 'global linkedness' is not a generic property even in $\mathbb{R}^2$. There are examples of pairs of vertices in rigid graphs which are globally linked in one generic realization, but not in another.

Given a graph $G=(V,E)$ and distinct vertices $x_1,x_2,\dots,x_{d+1}\in V$ with $x_1x_2\in E$, a *d-dimensional* 1*-extension* of G is a graph obtained from G by deleting the edge x_1x_2 and adding a new vertex z and new edges $zx_1,zx_2,\dots,zx_{d+1}$.

We first show that global linkedness is preserved by the 1-extension operation.

Theorem 2.1. *[9, for $d=2$] Let $H=(V,E)$ be a graph and let G be the d-dimensional 1-extension of H on some $v_1v_2\in E$. Suppose that $H-v_1v_2$ is rigid in $\mathbb{R}^d$ and that $\{x,y\}$ is globally linked in H in $\mathbb{R}^d$. Then $\{x,y\}$ is globally linked in G in $\mathbb{R}^d$.*

By using Theorem 1.1, we deduce that global rigidity is preserved by the 1-extension operation.

Corollary 2.2. *Suppose that H is globally rigid in $\mathbb{R}^d$ with $|V(H)|\geq d+2$ and G is obtained from H by a d-dimensional* 1*-extension. Then G is globally rigid in $\mathbb{R}^d$.*

In the rest of the section we will consider the $d=2$ case. Given a graph $G=(V,E)$, a subgraph $H=(W,C)$ is said to be an *M-circuit* in G if C is a circuit (i.e. a minimal dependent set) in $\mathcal{R}(G)$. In particular, G is an *M-circuit* if E is a circuit in $\mathcal{R}(G)$. For example, K_4, $K_{3,3}$ plus an edge, and $K_{3,4}$ are all M-circuits. Note that a graph G is redundantly rigid if and only if G is rigid and each edge of G belongs to a circuit in $\mathcal{R}(G)$ i.e. an M-circuit of G.

Given a matroid $\mathcal{M} = (E, \mathcal{I})$, we define a relation on E by saying that $e, f \in E$ are related if $e = f$ or if there is a circuit C in $\mathcal{M}$ with $e, f \in C$. It is well-known that this is an equivalence relation. The equivalence classes are called the *components* of $\mathcal{M}$. If $\mathcal{M}$ has at least two elements and only one component then $\mathcal{M}$ is said to be *connected*.

We say that a graph $G = (V, E)$ is *M-connected* if $\mathcal{R}(G)$ is connected. Thus M-circuits are special M-connected graphs. Another example is the complete bipartite graph $K_{3,m}$, which is M-connected for all $m \geq 4$. The *M-components* of G are the subgraphs of G induced by the components of $\mathcal{R}(G)$. Note that the M-components of G are induced subgraphs.

One of the main results of this section is the characterization of globally linked pairs in M-connected graphs.

Theorem 2.3. *[9] Let $G = (V, E)$ be an M-connected graph and $x, y \in V$. Then $\{x, y\}$ is globally linked in G if and only if there are three pairwise openly disjoint xy-paths in G.*

Based on this result, one can formulate the following Conjecture for the general case. Note that the 'if' part of this Conjecture is a direct corollary of Theorem 2.3.

Conjecture 2.4. *The pair $\{x, y\}$ is globally linked in a graph $G = (V, E)$ if and only if either $xy \in E$ or there is an M-component H of G with $\{x, y\} \subseteq V(H)$ such that there are three pairwise openly disjoint xy-paths in H.*

This conjecture would give rise to a polynomial algorithm to determine when a pair of vertices is globally linked in a graph. Given a graph $G = (V, E)$, [1] gives an algorithm which determines the M-components of G in $O(|V|^2)$ time. We can also determine whether two vertices of G are joined by three openly disjoint paths in $O(|V| + |E|)$ time, see [12].

The other main result of this section is the characterization of globally linked pairs of vertices in minimally rigid graphs, which is based on the following two Theorems about 1-extensions and non-globally-linked pairs of vertices.

Theorem 2.5. *Let $H = (V, E)$ be a rigid graph and let G be a 1-extension of H on some edge $uw \in E$. Then $\{u, w\}$ is globally linked in G if and only if $H - uw$ is rigid.*

Theorem 2.6. *Let $H = (V, E)$ be a rigid graph and let G be a 1-extension of H on some edge $uw \in E$. Suppose that $H - uw$ is not rigid and that $\{x, y\}$ is not globally linked in H for some $x, y \in V$. Then $\{x, y\}$ is not globally linked in G.*

Theorem 2.7. *Let $G = (V, E)$ be a minimally rigid graph and suppose that $xy \notin E$. Then $\{x, y\}$ is not globally linked.*

Since the only M-components of a minimally rigid graph are subgraphs containing single edges, Theorem 2.7 implies that Conjecture 2.4 holds for minimally rigid graphs.

The theory of globally rigid graphs can be applied in localization problems of sensor networks, see for example [3]. A generalization of global rigidity, unique localizability, also has direct applications in sensor network localization, see [5]. Let (G, p) be a generic framework with a designated set $P \subseteq V(G)$ of vertices. We say that a vertex $v \in V(G)$ is *uniquely localizable* in (G, p) with respect to P if whenever (G, q) is equivalent to (G, p) and $p(b) = q(b)$ for all vertices $b \in P$, then we also have $p(v) = q(v)$. We can think of P as the set of *pinned vertices* (or *anchor nodes* in a sensor network). We call a vertex v *uniquely localizable* in graph G, with respect to $P \subseteq V(G)$, if v is uniquely localizable with respect to P in all generic frameworks (G, p). For a graph G and a set $P \subseteq V(G)$ let $G + K(P)$ denote the graph obtained from G by adding all edges bb' for which $bb' \notin E$ and $b, b' \in P$. Using Theorem 2.3 we can derive the following characterization of uniquely localizable vertices when $G + K(P)$ is M-connected.

Theorem 2.8. *[9] Let $G = (V, E)$ be a graph, $P \subseteq V$ and $v \in V - P$. Suppose that $G + K(P)$ is M-connected. Then v is uniquely localizable in G with respect to P if and only if $|P| \geq 3$ and $\kappa(v, b) \geq 3$ for all $b \in P$.*

Another interesting application of Theorem 2.3 is that we can determine the number of non-congruent generic realizations of M-connected graphs. Given a rigid generic framework (G, p), let $h(G, p)$ denote the number of distinct congruence classes of frameworks which are equivalent to (G, p) (it is easy to see that this number is finite). Given a rigid graph G, let $h(G) = \max\{h(G, p)\}$, where the maximum is taken over all generic frameworks (G, p). For a graph $G = (V, E)$ and $u, v \in V$, let $b(u, v)$ denote the number of components of $G - \{u, v\}$ and put $c(G) = \sum_{u,v \in V}(b(u, v) - 1)$.

Theorem 2.9. *[9] Let G be an M-connected graph. Then $h(G, p) = 2^{c(G)}$ for all generic realizations (G, p) of G.*

3 Globally rigid frameworks

The results of this section are based on [11]. In this section we are concerned with the following algorithmic problem: given a graph G, how to create, in polynomial time, a globally rigid realization (G, p) in $\mathbb{R}^d$, if such a realization exists? In the dissertation we develop an algorithm for the case when $d = 2$ and G is globally rigid.

One of the difficulties is due to the fact that the output of the algorithm, which is a realization of G with rational coordinates, is non-generic. However, there is no 'simple' sufficient condition for the global rigidity of a non-generic framework. The algorithm is based on a sufficient condition for global rigidity which is based on stress matrices.

Another issue is the level of degeneracy of the framework (G, p) output by the algorithm. Since rather degenerate frameworks may be globally rigid (for example, if G is connected and all vertices are mapped to the same point), it is natural to impose certain additional requirements. It is known, see e.g. [2], that if (G, p) is a globally rigid and infinitesimally rigid framework then there exists an $\epsilon > 0$ such that if $||p(v) - q(v)|| < \epsilon$ for all $v \in V$ then (G, q) is also globally rigid. Thus infinitesimal rigidity makes the framework 'stable' in terms of global rigidity. Therefore it is natural to try to make (G, p) infinitesimally rigid as well. Given a graph $G = (V, E)$ we say that a 1-extension on the edge uw and vertex t is a *triangle-split* if $\{ut, wt\} \subseteq E$ (that is, if u, w, t induce a triangle of G). A graph will be called *triangle-reducible* if it can be obtained from K_4 by a sequence of triangle-splits. We note that triangle-reducible graphs are 3-connected redundantly rigid planar graphs with $2|V| - 2$ edges.

Theorem 3.1. *[11] Let $G = (V, E)$ be a globally rigid graph on at least four vertices. Then one can construct, in polynomial time, a globally rigid realization (G, p), where $p(V)$ spans $\mathbb{R}^2$. Furthermore, if G is triangle-reducible, the constructed realization can be chosen to be infinitesimally rigid, too.*

4 Rigidity of tensegrity graphs

The results of this section are based on [10]. A *tensegrity graph* $T = (V; B \cup C \cup S)$ is a simple graph on vertex set $V = \{v_1, v_2 \ldots, v_n\}$ whose edge set is partitioned into three pairwise disjoint sets B, C, and S, called *bars, cables*, and *struts*, respectively. The elements of $E = B \cup C \cup S$ are the *members* of T. A tensegrity graph containing no bars is called a *cable-strut tensegrity graph.* The *underlying graph* of T is the (unlabeled) graph $\overline{T} = (V; E)$. A d-dimensional *tensegrity framework* is a pair (T, p), where T is a tensegrity graph and p is a map from V to $\mathbb{R}^d$. (T, p) is also called a *realization* of T.

An *infinitesimal motion* of a tensegrity framework (T, p) is an assignment of infinitesimal velocities $q : V \to \mathbb{R}^d$ to the vertices, such that

$$
\begin{aligned}
(p(u) - p(v))(q(u) - q(v)) &= 0 \quad \text{for all } uv \in B, \\
(p(u) - p(v))(q(u) - q(v)) &\leq 0 \quad \text{for all } uv \in C, \\
(p(u) - p(v))(q(u) - q(v)) &\geq 0 \quad \text{for all } uv \in S.
\end{aligned}
$$

An infinitesimal motion is *trivial* if it can be obtained as the derivative of a rigid congruence of all of $\mathbb{R}^d$ restricted to the vertices of (T, p). The tensegrity framework (T, p) is *infinitesimally rigid* in $\mathbb{R}^d$ if all of its infinitesimal motions are trivial. A tensegrity graph T is said to be *rigid* in $\mathbb{R}^d$ if it has an infinitesimally rigid realization (T, p) in $\mathbb{R}^d$.

In this section we consider two combinatorial problems related to tensegrity graphs in the plane: (1) Given a graph $G = (V, E)$, how to find a cable-strut labeling $E = C \cup S$ of the edges for which the resulting tensegrity graph $T = (V; C \cup S)$ is rigid in $\mathbb{R}^2$. (Note that G has such a rigid cable-strut labeling if and only if G is redundantly rigid in $\mathbb{R}^2$.) (2) Given a cable-strut tensegrity graph $T = (V; C \cup S)$, decide whether T is rigid in $\mathbb{R}^2$. Our main result for the first problem is an efficient combinatorial algorithm for finding a rigid cable-strut labeling, if it exists. In the second part of this section we give a characterization for rigid tensegrity graphs in the plane where the underlying graph is either the complete graph or the wheel graph.

Both of these results will use the 'labeled generalizations' of the 1-extension operation. Let $T = (V; B \cup C \cup S)$ be a tensegrity graph, let $uw \in C \cup S$ be a cable or strut of T and let $t \in V - \{u, w\}$ be a vertex. The *labeled* 1-*extension* operation deletes the member uw, adds a new vertex v and new members vu, vw, vt, satisfying the condition that if uw is a cable then at least one of vu, vw is not a strut, and if uw is a strut then at least one of vu, vw is not a cable. The new member vt may be arbitrary.

Lemma 4.1. *[10] Let T be a rigid tensegrity graph and let T' be a tensegrity graph obtained from T by a labeled 1-extension. Then T' is also rigid.*

Using this extension lemma and another similar one for 'gluing together' tensegrity graphs along an edge, the algorithm to find a rigid cable-strut labeling is based on a new inductive construction of redundant graphs. We say that G is *redundant* if it has at least one edge and each edge of G is in an M-circuit.

Theorem 4.2. *[10] Let $G = (V, E)$ be a redundantly rigid graph in $\mathbb{R}^2$. Then the edge set of G has a cable-strut labeling $E = C \cup S$ for which the tensegrity graph $T = (V; C \cup S)$ is rigid. Furthermore, such a cable-strut labeling of E can be found in polynomial time.*

In the second part of this section, we solve the characterization of rigid tensegrity graphs, where the underlying graph is either a complete graph or a wheel graph.

Theorem 4.3. *Let $T = (V; C \cup S)$ be a cable-strut tensegrity graph on K_n for some $n \geq 5$. T is rigid in $\mathbb{R}^2$ if and only if $|C| \geq 3$ and $|S| \geq 3$ or there are four distinct vertices $u, v, w, t \in V$ such that $C = \{uv, wt\}$ or $S = \{uv, wt\}$.*

The wheel graph on n vertices is defined as $W_n = C_{n-1} + v_0 + \{v_0 v \mid v \in V(C_{n-1})\}$. It consists of a cycle C_{n-1} plus one *central vertex* v_0 and $n-1$ *central edges* $v_0 v$ for each $v \in V(C_{n-1})$. The vertices and edges of C_{n-1} are called *side vertices* and *side edges*, respectively. Given three adjacent side vertices u, v, w of W_n, a *forbidden* 3*-path* of W_n is defined as either $uvwv_0$ or uv_0vw.

Theorem 4.4. *Let $T = (V; C \cup S)$ be a cable-strut tensegrity graph on W_n for some $n \geq 6$ and $|S| \geq |C|$. T is rigid in $\mathbb{R}^2$ if and only if $|C| \geq 4$, or $|C| = 3$ and the cables do not form a forbidden* 3*-path, or $C = \{v_0 v, uw\}$ where u, v, w are distinct side vertices.*

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